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A METHOD OF APPROXIMATION.

By S. A. COREY, Hiteman, Iowa.

The following method of approximation may or may not be new, but as I believe it to be of practical importance I wish to call attention to it, and to point out that to secure the same degree of accuracy, it involves, at least in some cases, less labor than does the method of mechanical quadrature. It can also be used in certain cases where the method of mechanical quadrature fails, because the finite differences used in the latter method cannot always be found between the proper limits. On account of its rapid convergence it can also be used when other common methods fail.

We have the formula,*

$$f(x)=f(0)+\frac{x}{2m}\left\{f(x)+f(0)+2\left[f\left(\frac{x}{m}\right)+f'\left(\frac{2x}{m}\right)+\dots+f\left(\frac{m-1}{m}x\right)\right]\right\}$$

$$-\frac{B_1x^2}{m^2.2!}[f'(x)-f'(0)]+\frac{B_2x^4}{m^4.4!}[f^{iv}(x)-f^{iv}(0)]-\frac{B_3x^6}{m^6.6!}[f^{vi}(x)-f^{vi}(0)]+\dots.$$

$$+(-1)^n\frac{B_nx^{2n}}{m^{2n}.\,(2n)!}[f^{2n}(x)-f^{2n}(0)]+\dots \quad \dots(1),$$

(B_1, B_2, B_3, etc., being Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$, $\frac{5}{66}$, etc.)

For brevity, (1) may be written,

$$f(x)=F(x,m)+\frac{s_2}{m^2}+\frac{s_4}{m^4}+\frac{s_6}{m^6}+\text{ etc. }\dots(2),$$

in which s_2, s_4, s_6, etc., are independent of m. It is evident that m may be considered a convergence factor which makes $F(x, m)$ approach $f(x)$ as m approaches infinity, F and f becoming equal as m becomes infinite. But as m increases, the labor of evaluating $F(x, m)$ also increases, so that, in practice, it is found desirable to obtain one or more of the higher derivatives involved in s_2, s_4, s_6, etc., in order to permit the use of a smaller value of m, and yet to attain the required degree of accuracy. In certain cases it becomes exceedingly difficult to obtain the form and value of the higher derivatives involved in s_2, s_4, s_6, etc., and chiefly for this reason, it becomes desirable to eliminate from (2) certain of these quantities, (s_2, s_4, s_6, etc.), after a certain degree of approximation has been reached. To accomplish this elimination take more than one value of m and set down a separate equation for each value chosen, thus,

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*See *Annals of Mathematics*, Second Series, Vol. 5, No. 4, July, 1904.

$$f(x)=F(x,\ m_1)+\frac{s_2}{m_1^2}+\frac{s_4}{m_1^4}+\frac{s_6}{m_1^6}+\dots\ \dots(3),$$

$$f(x)=F(x,\ m_2)+\frac{s_2}{m_2^2}+\frac{s_4}{m_2^4}+\frac{s_6}{m_2^6}+\dots\ \dots(4),$$

$$f(x)=F(x,\ m_3)+\frac{s_2}{m_3^2}+\frac{s_4}{m_3^4}+\frac{s_6}{m_3^6}+\dots\ \dots(5),$$

etc. (In selecting the values of m it is usually convenient to take the highest value such that one or more of the lesser values will be sub-multiples thereof.)

It is evident that from $(r+1)$ such linear equations there may be eliminated r of the quantities s, say s_{2i}, $s_{2(i+1)}$, ..., $s_{2(i+r-1)}$. There results,

$$f(x)=\frac{\begin{vmatrix} M_1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \dots & m_1^{-2(i+r-1)} \\ M_2 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \dots & m_2^{-2(i+r-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ M_{(r+1)} & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \dots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}}{\begin{vmatrix} 1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \dots & m_1^{-2(i+r-1)} \\ 1 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \dots & m_2^{-2(i+r-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \dots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}}\ \dots(6),^*$$

where M_1, M_2, ..., $M_{(r+1)}$, represent the approximate value of $f(x)$ obtained by (2) for m_1, m_2, ..., $m_{(r+1)}$, respectively, by the use of all the terms preceding s_{2i}/m^{2i}.

If all derivatives higher than the $(2i+2r-1)$th are zero, or if all such *even-numbered* derivatives are equal for $x=x$ and $x=0$, (6) gives the exact value of $f(x)$. In other cases the degree of accuracy attained will be no less than would have been obtained by taking the smallest value of m used in (6) and developing $f(x)$ by (2) so far as to include all the r terms eliminated in (6).

As a special case of (6), let $r=2$, $i=2$. Here $M=F(x,\ m)+\frac{s_2}{m^2}$, and we get after reducing,

$$f(x)=\frac{m_1^6(m_2^2-m_3^2)M_1+m_2^6(m_3^2-m_1^2)M_2+m_3^6(m_1^2-m_2^2)M_3}{m_1^6(m_2^2-m_3^2)+m_2^6(m_3^2-m_1^2)+m_3^6(m_1^2-m_2^2)}\dots(7).$$

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*Determinants of this form may be readily evaluated. See Weld's *Theory of Determinants*, Articles 23 and 27.

It is clear that the more nearly all the M's approach $f(x)$, the more nearly does (6) give the exact value of $f(x)$, and that this increase in approximation is obtained by sufficiently increasing the integers, m, i, and r, or any of them.

The following points may be noted in passing:

First. After choosing certain values of m, r, and i, $f(x)$ is developed by (6) in linear terms of the $(r+1)$ quantities $(M_1, M_2, ..., M_{(r+1)})$, each M being an approximate value of $f(x)$.

Second. If $M=F(x, m)$ and if the values (but not the form) of $f'(x)$ be given at each of the $(m+1)$ equidistant points $(0, x/m, 2x/m, ..., x)$ for each of the $(r+1)$ values of m chosen, the value of $f(x)$ may be very closely approximated, although in such case the law of its development would not be expressly stated, but it must be assumed that some such law exists in order that $f(x)$ may be developable in terms of x. This makes the method of practical value in many scientific problems where it becomes necessary to find $f(x)$ approximately, and when the only available data are a number of observations of the values of $f'(x)$ at intervals of x/m, the approximation being closer in this method than in that of mechanical quadrature, especially where the intervals (x/m) are few.

A discussion of the cases where (6) does not give the value of $f(x)$ is not here entered into, but the following considerations will be pertinent:

First. As (6) is obtained from a number of equations, (3), (4), (5), etc., care should be taken that each of these equations actually does give a closer approximation of $f(x)$, when the number of terms employed include those eliminated in deriving (6), than does M.

Second. As (6), (2), and (1) are all dependent on the sum of a number of series developed by Stirling's formula, it becomes necessary that each of these underlying Stirling's series should give a true and convergent development of f between any and all of the adjacent, equidistant points (x/m) at which the value of f' is taken.

The following example has been chosen because its development by formula (1) has been given in the MONTHLY,* Vol. XIII, No. 4, and because its solution by mechanical quadrature has been given by Dr. G. W. Hill in the *Analyst*, Vol. II, p. 120, 1875.†

Example. Evaluate $\int_0^{\frac{1}{2}\pi}\frac{xdx}{\sin x(1+.16\cos^2x)^{\frac{3}{2}}}$.

Taking $r=2$, $i=2$, $m_1=6$, $m_2=3$, $m_3=2$, we get $M_1=1.657,626,355$, $M_2=1.657,490,406$, $M_3=1.657,013,853$. Substituting in (7), we get,

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*The value of the integral is incorrectly given in the MONTHLY as, 1.657,636,524, instead of 1.657,636,257. The error is due to the fact that

$$f^{vi},\left[\frac{2596\pi^6}{4200.12^6.6!}\right]$$

is carried out as .000,000,009 instead of .000,000,276.

†See also Hill's *Collected Works*, Vol. I, p. 204.

$$f(\tfrac{1}{2}\pi)-f(0)=\int_0^{\frac{1}{2}\pi}\frac{xdx}{\sin x(1+.16\cos^2x)^{\frac{3}{2}}}=\frac{1728M_3-23328M_2+233280M_1}{211{,}680}$$

$$=1.657{,}636{,}33\ldots(8).$$

We may check the accuracy of the work of computation and determine the degree of approximation attained in (8) by finding the values of $f'(\frac{\pi}{8})$ and $f'(\frac{3\pi}{8})$. This done we may take $m_1=6$, $m_3=4$, $m_2=3$. Substituting these values of m_1, m_2, m_3, in (7) we get an expression similar to (8) which gives a value of the definite integral coinciding with the value found in (8) for six decimal places, thus proving the accuracy of the work and showing the number of decimal places to which result found by (8) is correct.

This result is not as accurate as that obtained in the April MONTHLY for the reason that the *smallest* value of m here used is 2. This is therefore a more accurate result than would have been obtained by taking $m=2$ and developing by (1) so far as to include the term involving B_4, but a less accurate result than was obtained in the MONTHLY by taking $m=6$ and developing by (1) far enough to include the term involving B_3. The above result is, however, as accurate as Dr. Hill's and was obtained with much less labor. To obtain a more accurate result than that given in the MONTHLY without finding any higher derivatives than are there given, take $i=4$, $r=1$, $m_1=6$, $m_2=3$, and substitute in (6). By using nine decimals throughout the result is found to be 1.657,636,259. By using ten decimals throughout a result correct to nine or ten decimal places would have been obtained.

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APPROXIMATION OF THE GREATEST ROOT OF A CUBIC EQUATION WITH THREE REAL ROOTS.

By CHARLES GILPIN, JR., Philadelphia, Pa.

We are concerned with the "irreducible case," in which Cardan's formula is of no value for computation. By replacing x by $-x$ if necessary, we need consider only the form

$$x^3-ax-b=0\quad(a\text{ and }b\text{ positive})\quad\ldots(1),$$

the two lesser roots of which are negative and the greatest root, which equals the sum of the lesser ones, is positive. It can be shown* that the greatest root g lies between the limits $\sqrt{a}$ and $\sqrt{(4a/3)}$. From equation (1) we obtain

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*$g>\sqrt{a}$ by (2). Since $b^2/4-a^3/27<0$ in the irreducible case, x^3-ax-b is positive for $x=\sqrt{(4a/3)}$, negative for $x=\sqrt{a}$. EDITOR.